

Two distinct classes of bound entanglement: PPT-bound and “multi-particle”-bound

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We introduce systematically with the help of Weyl operators novel classes of multipartite and multidimensional states which are all bound entangled for arbitrary dimension. We find that the entanglement is bound due to different reasons: unlockable due to the multi-particle nature and some states are in addition bound due the fact being positive under partial transposition (PPT). By a general construction (\mathcal{W} simplices) we obtain classes of states which have the same geometry concerning separability and entanglement independent of the number of involved particle pairs. Moreover, we introduce a distillation protocol and demonstrate for $d = 3$ that for a certain set of states the entanglement can be increased only up to a certain amount.

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Quantum entanglement is a key feature of quantum theory with many important consequences for modern physics. It has become a highly valuable resource for novel applications, such as cryptography and a formidable quantum computer. However, the mathematical and/or physical characterization of all types of entanglement and their implementations are far from being fully explored. E.g. the quantification or even the classification of entanglement of multipartite systems is still an open problem.

This paper will analyze the nature of at least two distinct classes of bound entanglement, i.e. entanglement which cannot be distilled by local operations and classical communication (LOCC) into pure maximally entangled states, when each local observer possesses only one particle. This in return means that there should exist different applications for these states due to the different nature of their entanglement.

We first review a huge class of bipartite qudit states. A qudit is a quantum system with d degrees of freedom. With the help of group theoretical methods which allows for considerable simplifications a geometrical picture of the state space can be drawn, i.e. the properties separability, *bound* entanglement or PPT entanglement (PPT = positive under partial transposition) and NPT entanglement (NPT = negative under partial transposition) can be characterized. For bipartite qudits this state space was called “magic” simplex \mathcal{W} in Ref. [1] and extensively discussed in Refs. [2, 3] in different contexts. The construction of a simplex of states with maximally mixed subsystems has so far proven to be a powerful tool in analyzing bipartite qubits and qutrits (e.g. Ref. [4, 5, 6]) and recently even for multipartite qubits [7]. It provides a deep insight into the structure of entangled states and helps in constructing entanglement witnesses and exploring entanglement measures.

We will extend the simplex of bipartite qudits, i.e. one pair of qudits, to n pairs of qudits where n is any natural number. We will prove that interestingly this extended class has the same properties concerning separability, bound entanglement and NPT-entanglement by proving

that the optimal entanglement witnesses reduces to the same mathematical conditions (Theorem 2). Therefore, results for bipartite qudits become automatically true for any n pairs of bipartite qudits, which may otherwise due to the high computational effort would not be obtainable.

This extended class of states shows due to their multi-particle nature a feature which was called unlockable-bound entanglement [8, 10]. In detail *NPT*-entangled states can be distilled to certain extremal states, the so called “vertex” states of the simplex $\mathcal{W}^{\otimes n}$, however, not into pure maximally entangled states: this novel class of states are bound to their own class. For multipartite qubits this was shown in Refs. [7]. We prove in this paper that this is a general feature of such multipartite simplex states and, moreover, the fact that *PPT*-bound entangled states exist for dimensions $d \geq 3$ implies that there are two different kinds of bound entanglement. Explicitly, we give a multidimensional distillation protocol for $d = 3$ which distills certain states within the simplex to the vertex states, which are themselves bound entangled.

The magic simplex \mathcal{W} for bipartite qudits: For bipartite qudits the vertex states $P_{i,j}$ of the “magic” simplex \mathcal{W} are the maximally entangled states in d dimensions (Refs. [1, 2]):

$$|\Phi^+\rangle := \sum_{i=0}^{d-1} |ii\rangle, \quad P_{0,0} := |\Phi^+\rangle\langle\Phi^+| \quad (1)$$

$$P_{k,l} := \mathbb{1}_d \otimes W_{k,l} P_{0,0} \mathbb{1}_d \otimes W_{k,l}^\dagger \quad (2)$$

where the $W_{k,l}$ are the Weyl operators defined by

$$W_{k,l}|s\rangle = w^{k(s-l)}|s-l\rangle \quad \text{with} \quad w = e^{2\pi i/d}. \quad (3)$$

The magic simplex \mathcal{W} is the convex combination of all vertex states

$$\mathcal{W} := \left\{ \sum_{k,l=0}^{d-1} c_{k,l} P_{k,l} \mid c_{k,l} \geq 0, \sum_{k,l=0}^{d-1} c_{k,l} = 1 \right\}. \quad (4)$$

One main property of this class of states forming a $d^2 - 1$ dimensional simplex is that any trace of one particle

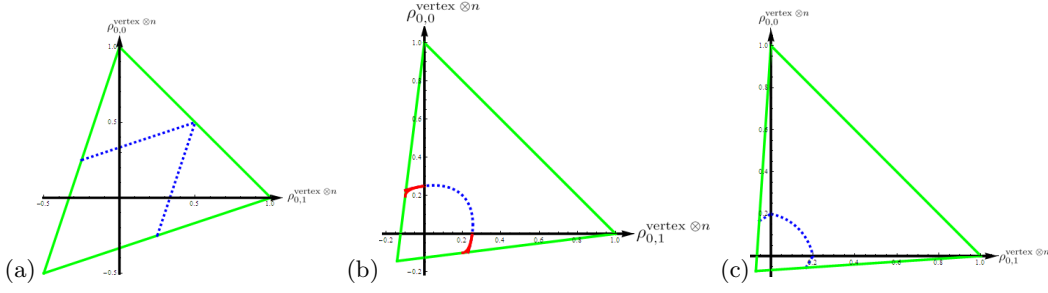


FIG. 1: (Color online) Slices via the simplices for the states which are mixtures of any two vertex states and the maximally mixed state, Eq. (13), for the dimensions (a) $d = 2$, (b) $d = 3$, (c) $d = 4$. The (green) triangles are given by the positivity condition, the dotted (blue) lines/curves represents the PPT condition. For $d = 3$ one finds a whole region of PPT bound entanglement if either α or β is negative (filled (red) region). As expected the region of separable states shrinks with increasing dimension d .

results in a maximally mixed state. We want to conserve this property for the multipartite scenario, i.e. any trace of one or more particles should result into a maximally mixed state:

$$\begin{aligned} \rho_{0,0}^{\text{vertex} \otimes n} &:= \frac{1}{d^2} \sum_{i,j=0}^{d-1} P_{i,j} \otimes P_{i,j} \otimes \cdots \otimes P_{i,j} \\ &= \frac{1}{d^2} \sum_{i,j=0}^{d-1} P_{i,j}^{\otimes n}. \end{aligned} \quad (5)$$

For $d = 2$ this state was investigated by Smolin [8] and has proven to be an interesting state, exhibiting many counter-intuitive properties: such that it is biseparable under any bipartite cut, but ignorance of any arbitrary number of subsystems will render this state useless for quantum informational tasks, Refs. [9, 10], though it violates a Bell inequality (see Refs. [7, 9, 11]). Moreover, applying the two sets of multipartite entanglement measures proposed in Ref. [12] it turns out that n paired LOCCs are needed to prepare the state, whereas $2n$ parties are needed to cooperate locally to perform quantum informational tasks with that state.

We prove now that the above state has unlockable entanglement and then generalize to a whole class of states with all that features.

Theorem 1: The state, Eq. (5), is (multipartite) bound entangled for any dimension d , because no locally working of all involved parties can by LOCC distill a pure maximally entangled state.

Proof. As for every state that exhibits a partial separability like $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$ will remain A - B separable under every LOCC of the form:

$$\Lambda_{LOCC}[\rho] = \frac{\sum_k A_k \otimes B_k \rho A_k^\dagger \otimes B_k^\dagger}{\text{Tr}(\sum_k A_k \otimes B_k \rho A_k^\dagger \otimes B_k^\dagger)} \quad (6)$$

and the state in question allows a biseparable decomposition even if two subsystems are arbitrarily exchanged,

this special property is preserved under LOCC. No maximally entangled pure state can exhibit this property, hence the state is bound entangled. \square

The magic simplex $\mathcal{W}^{\otimes n}$ for n pairs of qudits:

A certain vertex states of any n pairs of qudits can be defined by

$$\begin{aligned} n = 1 : \quad \rho_{0,0}^{\text{vertex} \otimes 1} &:= P_{0,0} \\ n \geq 2 : \quad \rho_{0,0}^{\text{vertex} \otimes n} &:= \frac{1}{d^2} \sum_{i,j=0}^{d-1} P_{i,j}^{\otimes n}. \end{aligned} \quad (7)$$

By applying in one subsystem a Weyl operator $W_{k,l} := \mathbb{1}_d \otimes W_{k,l}$ one obtains as before the remaining $d^2 - 1$ vertex states

$$\begin{aligned} \rho_{k,l}^{\text{vertex} \otimes n} &= \mathbb{1}_{d^2}^{\otimes(n-1)} \otimes W_{k,l} \rho_{0,0}^{\text{vertex} \otimes n} \mathbb{1}_{d^2}^{\otimes(n-1)} \otimes W_{k,l}^\dagger \\ &= \frac{1}{d^2} \sum_{i,j=0}^{d-1} P_{i,j}^{\otimes n-1} \otimes W_{k,l} P_{i,j} W_{k,l}^\dagger. \end{aligned} \quad (8)$$

Note that if the Weyl operator is applied on a different subsystem we obtain an equivalent simplex, however, with different labeling (all states and partial states have for any n same eigenvalues).

Now we can define a huge class of states which have the same geometry concerning separability and entanglement for a given d , the “magic” n pair qudit simplex $\mathcal{W}^{\otimes n}$:

$$\mathcal{W}^{\otimes n} := \left\{ \sum_{k,l=0}^{d-1} c_{k,l} \rho_{k,l}^{\text{vertex} \otimes n} \mid c_{k,l} \geq 0, \sum_{k,l=0}^{d-1} c_{k,l} = 1 \right\}.$$

These states have the same properties as the vertex states, i.e. all subsystems are maximally mixed, all states have n -separable decompositions, where always any two subsystems can be grouped together and single subsystems may arbitrarily be interchanged. The mixedness of any vertex state, $M := \frac{d}{d-1} (1 - \text{Tr}(\rho_{k,l}^{\text{vertex} \otimes n} \rho_{k,l}^{\text{vertex} \otimes n}))$,

for $n \geq 2$ is $\frac{1-d^{-2}}{1-d^{-n}}$, thus gets less mixed with increasing n and/or d .

We prove now that the structure of separability is for any n equivalent by the powerful tool of witnesses, then we proceed to discuss the feature of bound entanglement and unlockable-bound entanglement.

Optimal witnesses in the simplex $\mathcal{W}^{\otimes n}$: An entanglement witness EW_ρ is a criterion to “witness” for an certain state ρ that it is not in the set of separable states SEP . Knowing that SEP is convex it can be completely characterized by the tangential hyperplanes, thus we search for tangential or optimal witnesses on the surface of SEP , i.e.

$$EW_\rho^{opt} = \{K = K^\dagger \neq 0 | \forall \rho_{sep} \in SEP : \text{Tr}(K \rho_{sep}) < 0 \text{ and } \text{Tr}(K \rho) = 0\}. \quad (9)$$

As proven in Ref. [1] any witness operator for states

within the simplex \mathcal{W} can only be of the form $K = \sum_{k,l} \kappa_{k,l} P_{k,l}$. As \mathcal{W} and $\mathcal{W}^{\otimes n}$ have the same group symmetries by their construction via the Weyl operators (see Theorem 6 in Ref. [1]) any witness operator within $\mathcal{W}^{\otimes n}$ has to have the form $K_n = \sum_{k,l} \kappa_{k,l} \rho_{k,l}^{\text{vertex} \otimes n}$.

Theorem 2: The operator $K_n = \sum_{k,l} \kappa_{k,l} \rho_{k,l}^{\text{vertex} \otimes n}$ is an optimal entanglement witness if $\det M_\Phi = 0$ with $M_\Phi = \sum_{k,l} \kappa_{k,l} W_{k,l} |\Phi\rangle \langle \Phi| W_{k,l}^\dagger \geq 0 \quad \forall \Phi \in \mathbb{C}^d$. This means that the set of separable, PPT-entangled and NPT-entangled states have for any d and all n the same geometry because the $d \times d$ matrix M_Φ is identical.

Proof. Any separable state ρ_{sep} can be written as a convex combination of pure product states and therefore $\text{Tr}(K_n \rho_{sep}) \geq 0 \quad \forall \rho_{sep} \in SEP$ implies that

$$\langle K_n \rangle := \langle \eta_1, \chi_1 | \otimes \langle \eta_2, \chi_2 | \otimes \dots \langle \eta_n, \chi_n | K_n | \eta_1, \chi_1 \rangle \otimes | \eta_2, \chi_2 \rangle \otimes \dots | \eta_n, \chi_n \rangle \geq 0 \quad \forall \eta_1, \chi_1, \eta_2, \chi_2 \dots \eta_n, \chi_n \in \mathbb{C}^d.$$

By the observation that $P_{k,l} = \frac{1}{d} \sum_{s,t=0}^{d-1} W_{k,l} |ss\rangle \langle t,t| W_{k,l}^\dagger = \frac{1}{d} \sum_{s,t} W_{k,l} \otimes \mathbb{1}_d |ss\rangle \langle t,t| W_{k,l}^\dagger \otimes \mathbb{1}_d$ follows

$$\langle \eta_i, \chi_i | P_{k,l} | \eta_i, \chi_i \rangle = \frac{1}{d} \sum_{s,t} \langle \eta_i | W_{k,l} | s \rangle \langle \chi_i | s \rangle \langle t | \chi_i \rangle \langle t | W_{k,l}^\dagger | \eta_i \rangle = \frac{1}{d} \langle \eta_i | W_{k,l} | \phi_i \rangle \langle \phi_i | W_{k,l}^\dagger | \eta_i \rangle, \quad (10)$$

where we defined all $\phi_i \in \mathbb{C}^d$ as $|\phi_i\rangle = \sum_s \langle \chi_i | s \rangle |s\rangle$. Therefore, $P_{k,l}$ is obviously an entanglement witness, because

$$\frac{1}{d} \langle \eta_i | W_{k,l} | \phi_i \rangle \langle \phi_i | W_{k,l}^\dagger | \eta_i \rangle = \frac{1}{d} |\langle \eta_i | \tilde{\phi}_i \rangle|^2 \geq 0, \forall \eta_i, \tilde{\phi}_i \in \mathbb{C}^d. \quad (11)$$

The expectation value of the witness operator K_n in $(d \times d)^n$ reduces to an expectation value of $d \times d$ operators

$$\begin{aligned} \langle K_n \rangle &= \frac{1}{d^2} \frac{1}{d^n} \sum_{k,l} \kappa_{k,l} \sum_{g,h} \langle \eta_1 | W_{g,h} | \phi_1 \rangle \langle \phi_1 | W_{g,h}^\dagger | \eta_1 \rangle \cdot \dots \cdot \langle \eta_{n-1} | W_{g,h} | \phi_{n-1} \rangle \langle \phi_{n-1} | W_{g,h}^\dagger | \eta_{n-1} \rangle \\ &\quad \cdot \langle \eta_n | W_{k,l} | \phi_n \rangle \langle \phi_n | W_{k,l}^\dagger | \eta_n \rangle = \frac{d^2}{d^2} \cdot C \cdot \frac{1}{d^n} \sum_{k,l} \kappa_{kl} \langle \eta_n | W_{k,l} | \tilde{\phi}_n \rangle \langle \tilde{\phi}_n | W_{k,l}^\dagger | \eta_n \rangle \end{aligned} \quad (12)$$

with $C \geq 0$. Therefore, K_n is an entanglement witness if the operator $M_\phi = \sum_{k,l} \kappa_{k,l} W_{k,l} |\phi\rangle \langle \phi| W_{k,l}^\dagger$ is not negative for all $\phi \in \mathbb{C}^d$ and it is optimal if $\det M_\phi = 0$. \square

Example showing the geometry of separability and PPT-bound entanglement for different dimensions: Let us consider any two vertex states mixed with the totally mixed state, i.e.

$$\rho = \frac{1 - \alpha - \beta}{d^2} \mathbb{1}_d^{\otimes 2n} + \alpha \rho_{0,0}^{\text{vertex} \otimes n} + \beta \rho_{0,1}^{\text{vertex} \otimes n}. \quad (13)$$

The positivity condition of the density matrix on the parameters α, β give three lines which form a triangle.

Likewise we obtain the parameter region for the states which are PPT entangled. This is visualized in Fig. 1 for dimension $d = 2, 3, 4$. The authors of Ref. [1] found by optimizing the witness operator for bipartite qutrits PPT-bound entanglement if either α or β is negative. By Theorem 2 this means that we found a whole region of PPT-bound entanglement for any number of qutrit pairs n .

Distilling bound entanglement: While the basic geometric structure of separable and PPT-bound entangled and entangled states remains unchanged with n , the properties of the states in the simplex change drastically. They are bound entangled as the vertices states cannot

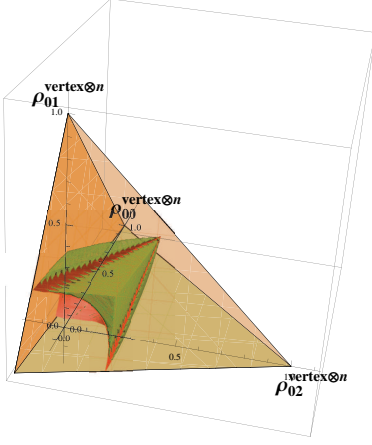


FIG. 2: Three dimensional slice through the eight dimensional simplex for $d = 3$, given by Eq. (14). The (transparent yellow) tetrahedron is given by the positivity condition, the (red) cone represents the PPT condition. Inside the PPT cone there are also bound entangled states, which cannot be distilled at all. The (green) Christmas tree shaped area is obtained via application of the distillation protocol and shows states that can not be distilled to one of the edge states. This area was obtained numerically, yet it is most intriguing that it, up to numerical precision, coincides with the states not detected by entanglement measures derived from the m -concurrence [12].

be distilled (theorem 1). However, as we prove in the following for $d = 3$ some states inside $\mathcal{W}^{\otimes n}$ can be distilled by a certain protocol to the vertices states. Let's consider the following distillation protocol:

1. Take a copy of the state: $\rho^{\otimes 2}$, the first dit will be regarded as source, the second as target dit.
2. Apply the unitary gate U_m in all subsystems: $\rho_T = U_m^{\otimes n} \rho^{\otimes 2} U_m^{\otimes n}$ with $U_m := (1 - \delta_{ij}) |ij\rangle\langle ij| + \delta_{ij} (|ij\rangle\langle im| + |im\rangle\langle ij|)$.
3. Project onto $|m\rangle\langle m|$ in all target systems: $\mathbb{1}_d \otimes |m\rangle\langle m| \rho_T \mathbb{1}_d \otimes |m\rangle\langle m|$
4. Discard target dits.

With this protocol it is possible to “distill” many NPT-entangled states in the simplex into a vertex state. Consider e.g. the following state

$$\rho = \frac{1 - \alpha - \beta - \gamma}{9} \mathbb{1}_3^{\otimes 2n} + \alpha \rho_{0,0}^{\text{vertex} \otimes n} + \beta \rho_{0,1}^{\text{vertex} \otimes n} + \gamma \rho_{0,2}^{\text{vertex} \otimes n}. \quad (14)$$

This is an example of a so called “line” state, where the same Weyl operator connects all vertex states. This is visualized in Fig. 2. Surprisingly, the “distillable” states are the ones which are detected by the bounds on the multipartite qudit measure introduced in Ref. [12].

Clearly, for $n = 1$ the vertex states are pure and therefore it is a genuine distillation protocol, however, for

$n \geq 2$ the vertex states are no longer pure, the protocol distills up to a certain degree of entanglement and purity. Note, that for $d = 2$ and $n = 2$ this has already proven to be very useful, as the vertex states can be used to reduce communication complexity and for remote information concentration for $2n$ parties [10].

Conclusion: We have introduced a whole new class of bound entangled states for arbitrary n pairs of qudits (d degrees of freedom), the extended simplex $\mathcal{W}^{\otimes n}$, and proven that all states are non-distillable. The very nature of their bound entanglement stems from the multipartite construction and may be unlocked if two parties work together. Inside the simplex ($d \geq 3$) there also exist states which cannot be distilled, because they are nonseparable PPT-states. Thus in the multipartite and multidimensional scenario there exist at least two classes of bound entangled states: those which may be unlocked via multipartite cooperation and those which cannot be distilled even if two or more parties cooperate. One could also say the PPT-bound states for any $n \geq 2$ are *bound-bound* entangled, i.e. PPT-bound and multi-particle-bound. Moreover, this feature is given for arbitrary dimensions d . In Fig. 1 we showed how the geometry of separability, PPT and entanglement changes with increasing dimension d . Last but not least our distillation protocol for $d = 3$ shows that almost all NPT-entangled states are two copy distillable to the vertex states and, consequently, the states are noise resistant. All these special features of these state spaces may help to develop novel applications and novel schemes for multipartite quantum communication.

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